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ON THE PERTURBATION FRONT STRUCTURE FOR TRANSPORT PROCESSES WITH SPATIAL–TEMPORAL NONLOCALITY

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The one-dimensional problem of the propagation of a perturbation front from a point instantaneous source for transport processes with spatial–temporal nonlocality is considered. A class of nonlocality kernels with a singularity of the form t^{-1} for small times is used. The front propagation speed v is calculated and an expression for perturbations in the vicinity of the front is derived in the form of an asymptotic series in powers of the parameter $\tau = t - xv^{-1}$.

Transport processes (heat conduction, diffusion, propagation of transverse modes in a viscous fluid, filtration, etc.) are usually described by parabolic equations of the heat equation type. A property of these equations is the infinite signal propagation speed. But this contradicts physical laws since the propagation speed of every signal should not at least exceed the speed of light in vacuum. Therefore, a physically justified description of transport processes requires modification of the basic dynamic equations.

The above circumstance has been discussed in the literature for a long time. Cattaneo [1] proposed to replace the parabolic equation by a hyperbolic equation with a small parameter at the higher derivative with respect to time. Later it was established that the method of [1] is a special case of the more general approach that takes into account the relaxational relationship between the flux of the quantity being transported and its gradient [2]. The question of the limiting speed of a signal was studied for a relaxation kernel of general form [3]. It was shown that in propagation of a signal from a point source, the perturbation front structure is determined by the behavior of the relaxation kernel at small times [4].

However, it is known that models with relaxation (i.e., models with temporal nonlocality) are approximations of more general models that take into account not only temporal but also spatial dispersion. Constitutive relations with spatial–temporal nonlocality arise in hydrodynamic description in kinetic theory and in classical and quantum statistical mechanics (see [5]). At the same time, the problem of the relationship between the condition of finiteness of the signal propagation speed and the properties of a kernel is more complicated in the case of models with spatial–temporal nonlocality. This problem is solved in the present paper for a class of kernels with a singularity of the form t^{-1} for small times. For simplicity, we consider the one-dimensional case.

Let the dynamics of the quantity $u = u(t, x)$ be described by the local conservation equation with sources

$$\partial_t u + \partial_x J = q(t, x), \tag{1}$$

where J is the flux of the quantity u and $q = q(t, x)$ is the source field. In various problems, the quantity u can describe temperature, concentration, the transverse velocity component in a viscous fluid, fluid pressure during filtration in a porous medium, etc. When the spatial–temporal nonlocality is taken into account, the dynamic equation (1) is closed by the constitutive relation

$$J = -\int K(t - t_1, x - x_1)\partial_x u(t_1, x_1)\, dt_1\, dx_1. \tag{2}$$

Substitution of (2) into (1) yields the integrodifferential equation for $u = u(t, x)$:

$$\partial_t u(t, x) - \int K(t - t_1, x - x_1)\partial_x^2 u(t_1, x_1)\, dt_1\, dx_1 = q(t, x). \tag{3}$$

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In the case of slow processes with a small spatial gradient, i.e., at the limit of low frequencies and long waves, relation (2) becomes

$$J = -æ\partial_x u, \qquad æ = \int K(t, x)\, dt\, dx,$$

and the dynamic equation (3) reduces to the ordinary heat equation

$$\partial_t u - æ\partial_x^2 u = q(t, x).$$

The kernel $K(t, x)$ characterizes the relaxation properties and spatial dispersion of the medium.

The function $K(t, x)$ should satisfy the following conditions that ensue from physical and thermodynamical considerations.

1. Since the properties of the medium are assumed to be invariant under the spatial inversion $x \to -x$, the following relation holds:

$$K(t, x) = K(t, -x). \tag{4}$$

2. For the Fourier image of the kernel

$$K_F(\omega, k) = \int \exp(-i(\omega t + kx))K(t, x)\, dt\, dx,$$

the dissipativity condition should hold or, in other formulation, the condition of consistency of the model with the second law of thermodynamics [5]

$$\operatorname{Re} K_F(\omega, k) \geqslant 0. \tag{5}$$

3. Since a class of models with finite speed of signal propagation is considered, the function $K(t, x)$ should vanish outside the cone

$$C_V = \{(t, x) \mid V^2t^2 - x^2 \geqslant 0,\ t \geqslant 0\}, \tag{6}$$

where V is a positive constant with the dimension of speed. It is known [6] that the condition that the support of the function $K(t, x)$ belongs to the cone (6) is equivalent to the holomorphy of the Fourier image $K_F(\omega, k)$ in the complex tube

$$T_V = \{(\omega, k) \mid V^{-2}(\operatorname{Im} \omega)^2 - (\operatorname{Im} k)^2 > 0,\ \operatorname{Im} \omega < 0\}. \tag{7}$$

4. Let us assume that inequality (5) holds not only for real values of the frequency ω and wave number k but also in the case where the frequency ω takes values in the lower complex half-plane $\operatorname{Im} \omega < 0$.

For simplicity, we shall use the system of units such that $V = 1$.

We consider the class of kernels of the form

$$K(t, x) = \theta(t - |x|) \exp(-\alpha t)[At^{-1} + \varphi(t, x)]. \tag{8}$$

Here θ is the Heaviside function, $A > 0$ and $\alpha > 0$ are constants, and $\varphi(t, x)$ is a function that is smooth and bounded in the cone C_1 and even in the spatial coordinate. Thus, the class of kernels considered is distinguished by a singularity of the form t^{-1} for small times.

The Fourier image of the kernel (8) can be represented as the sum

$$K_F(\omega, k) = \Phi_1(\omega, k) + \Phi_2(\omega, k),$$

where

$$\Phi_1(\omega, k) = A \int\limits_{t \geqslant |x|} \exp(-i((\omega - i\alpha)t + kx))t^{-1}\, dt\, dx = 2Ak^{-1} \int\limits_{t \geqslant 0} \exp(-i(\omega - i\alpha)t)t^{-1} \sin(kt)\, dt, \tag{9}$$

$$\Phi_2(\omega, k) = \int\limits_{t \geqslant |x|} \exp(-i((\omega - i\alpha)t + kx))\varphi(t, x)\, dt\, dx. \tag{10}$$

Integral (9) can be calculated in explicit form using formula 3.381.4 from [7]

$$\int\limits_{t \geqslant 0} t^{\nu - 1} \exp(-\mu t)\, dt = \mu^{-\nu}\Gamma(\nu), \qquad \operatorname{Re} \mu > 0, \quad \operatorname{Re} \nu > 0.$$

For this, we need to replace the power-type singularity t^{-1} in expression (9) by the singularity $t^{\nu-1}$ and then pass to the limit as $\nu \to 0$. As a result, we obtain

$$\Phi_1(\omega,k) = A(ik)^{-1}\ln((\omega+k-i\alpha)/(\omega-k-i\alpha)). \tag{11}$$

It is easy to check that for $\operatorname{Im} k = 0$ and $\operatorname{Im}\omega \leqslant 0$, we have

$$\operatorname{Re}\Phi_1(\omega,k) > 0. \tag{12}$$

Let us find the asymptotic behavior of the functions $\Phi_1(\omega,k)$ and $\Phi_2(\omega,k)$ as $\operatorname{Im}\omega \to -\infty$ in the complex tube T_1. From (11) we have

$$\Phi_1(\omega,k) = -2A(\operatorname{Im}\omega)^{-1} + O((\operatorname{Im}\omega)^{-2}). \tag{13}$$

Integrating by parts, for the function $\Phi_{2F}(\omega,k)$ we obtain

$$\Phi_2(\omega,k) = \zeta_1(\omega,k) + \zeta_2(\omega,k) + \zeta_3(\omega,k), \tag{14}$$

where

$$\zeta_1(\omega,k) = 2((i\omega+\alpha)^2+k^2)^{-1}\varphi(0,0),$$

$$\zeta_2(\omega,k) = (i\omega+\alpha)^{-1}\int\limits_0^{+\infty}((i\omega+\alpha+ik)^{-1}\exp(-i(\omega-i\alpha+k)x)$$

$$+\,(i\omega+\alpha-ik)^{-1}\exp(-i(\omega-i\alpha-k)x))\frac{d}{dx}\,\varphi(x,x)\,dx,$$

$$\zeta_3(\omega,k) = (i\omega+\alpha)^{-1}\int\limits_{t\geqslant|x|}\exp(-i((\omega-i\alpha)t+kx))\partial_t\varphi(t,x)\,dt\,dx.$$

The term $\zeta_3(\omega,k)$ in (14) has the functional form of the initial function (10) and can be represented in a form similar to (14). The term $\zeta_2(\omega,k)$ also admits integration by parts. Thus, the major contribution to (14) is from the term $\zeta_1(\omega,k)$ and the following estimate is valid:

$$\Phi_2(\omega,k) = O((\operatorname{Im}\omega)^{-2}). \tag{15}$$

Relations (12), (13), and (15) imply that condition 4 can be satisfied for a rather wide class of functions $\varphi(t,x)$ since the contribution of this function to the Fourier image of the kernel for $\operatorname{Im}\omega \to -\infty$ is an infinitesimal of higher order.

For Eq. (3), we consider the problem of the propagation of perturbations from a point instantaneous source $q(t,x) = a\delta(t)\delta(x)$ (a is an arbitrary number). Converting to the Fourier images in Eq. (3), we obtain

$$\Psi(\omega,k)u_F(\omega,k) = a, \qquad \Psi(\omega,k) = i\omega + k^2K_F(\omega,k). \tag{16}$$

Equation (16) enables us to find a solution for the quantity u in quadratures:

$$u(t,x) = a(2\pi)^{-2}\int\exp(i\omega t)\,d\omega\int\frac{\exp(ikx)}{\Psi(\omega,k)}\,dk.$$

Here the integral with respect to k is taken along the real axis, and the integral with respect to ω is taken along the straight line $\operatorname{Im}\omega = -\varepsilon$ (ε is a small positive number). By condition 4, the integrand is regular everywhere, and for large absolute values of k and ω, the integral is calculated in the sense of the principal value.

According to condition 3, the function $\Psi(\omega,k)$ is holomorphic in the tube T_1 [see (7)]. Formula (16) makes it possible to calculate the function $u_F(\omega,k)$ for complex values of k and ω. The perturbation propagation speed is defined as a maximum value $v \geqslant 1$ such that the function $u_F(\omega,k)$ is holomorphic in the tube T_V. Under this condition, the function $u(t,x)$ necessarily vanishes outside the cone C_V [6].

Equality (4) implies the relation $\Psi(\omega,k) = \Psi(\omega,-k)$. To solve the problem posed, it suffices to study the roots $k = k(\omega)$ of the equation

$$\Psi(\omega,k) = 0 \tag{17}$$

for the wave number with the additional condition $\operatorname{Im} k > 0$. We note that by condition 4, there are no solutions with $\operatorname{Im} k = 0$.

Let us give a numerical algorithm for finding the roots of Eq. (17) for fixed ω ($\operatorname{Im}\omega < 0$). The function $\Psi(\omega, k)$ is holomorphic with respect to the parameter k in the strip $|\operatorname{Im} k| < |\operatorname{Im}\omega|$. Therefore, generally speaking, Eq. (17) can have a discrete set of solutions $k_n = k_n(\omega)$ $(n = 1, 2, \dots)$.

We note that integration of the function (10) by parts yields

$$\Phi_2(\omega, k) = 2 \int\limits_{0\leqslant x\leqslant t} \exp(-i(\omega - i\alpha)t)\varphi(t, x)\cos(kx)\, dt\, dx$$

$$= \frac{2}{k^2}\,\varphi(0,0) + \frac{2}{k^2}\int\limits_{0\leqslant t} \exp(-i(\omega - i\alpha)t)\,\frac{d}{dt}\,\varphi(t,t)\cos(kt)\, dt$$

$$-\,\frac{2}{k}\int\limits_{0\leqslant x\leqslant t} \exp(-i(\omega - i\alpha)t)\partial_x\varphi(t, x)\sin(kx)\, dt\, dx.$$

The last term has the same functional structure as the function $\Phi_2(\omega, k)$ has and, consequently, it can also be transformed by integration by parts. As a result, for $|k| \to +\infty$, we have the asymptotic relation $|\Phi_2(\omega, k)| = O(|k|^{-2})$. Hence, with account of formula (11), we conclude that for $|k| \to +\infty$ the following asymptotic relation holds:

$$|\Psi(\omega, k)| = \pi A|k| + O(1).$$

Therefore, Eq. (17) can have solutions $k_n = k_n(\omega)$ only in a bounded domain of the complex plane.

Next, we find more precise positions of the roots of Eq. (17) on the complex plane. For this, we find the contour integrals

$$K_C = \frac{1}{2\pi i}\oint\limits_C \frac{\partial_k\Psi(\omega, k)}{\Psi(\omega, k)}\, dk,$$

where K_C is the number of roots of Eq. (17), with their multiplicities taken into account, in the domain enclosed by the contour C.

The algorithm for calculating the values of $k_n = k_n(\omega)$ with preassigned accuracy can be completed by Newton's iteration procedure.

The required value of the speed v is defined by the functions $k_n = k_n(\omega)$ through the relations

$$v^{-1} = \inf_{\omega,n}\,(H_n(\omega)), \qquad H_n(\omega) = \operatorname{Im} k_n(\omega)/(-\operatorname{Im}\omega). \tag{18}$$

We note that by definition, the infimum in (18) cannot be reached on the boundary of the domain of holomorphy. Neither can it be reached at an internal point in this domain. Indeed, if the infimum $v^{-1} = H_n(\omega_0)$ is reached at some point ω_0, then the harmonic function of two real variables ($\omega_1 = \operatorname{Re}\omega$ and $\omega_2 = \operatorname{Im}\omega$)

$$h(\omega_1, \omega_2) = \operatorname{Im} k_n(\omega_1 + i\omega_2) + v^{-1}\omega_2 \tag{19}$$

reaches the minimum equal to zero at the point $\omega_1 = \operatorname{Re}\omega_0$, $\omega_2 = \operatorname{Im}\omega_0$. By the properties of harmonic functions, the last is possible only if the function (19) is identical zero. Therefore, the solution depends linearly on the frequency, $k_n(\omega) = -v^{-1}\omega + \beta$, which contradicts Eq. (17) for $\omega \to 0$.

Thus, the infimum (18) can be reached only for $|\omega| \to +\infty$. Setting $z = -k/\omega$ and passing to the limit as $|\omega| \to +\infty$ in Eq. (17) with account of (11) and (14), in the main approximation we obtain the equation

$$1 + Az\ln((1 - z)/(1 + z)) = 0. \tag{20}$$

Equation (20) has a unique positive solution $z_0 = z_0(A) < 1$, which defines the required value of the perturbation propagation speed $v = z_0^{-1}$. The dependence of the speed v on the parameter A for small values of A is presented in Fig. 1. For large A, the approximate dependence $v \approx (2A)^{1/2}$ holds.

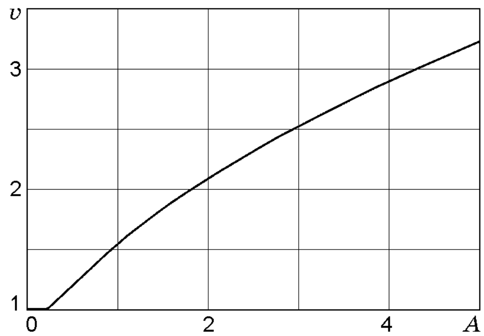

Fig. 1

Thus, the unique solution $k_1 = k_1(\omega)$ of Eq. (19) satisfies the condition $\operatorname{Im} k_1 > 0$. Here the function $k_1 = k_1(\omega)$ admits the asymptotic expansion into the series:

$$k_1(\omega) = -\omega \sum_{n=0}^{+\infty} z_n (i\omega)^{-n}. \tag{21}$$

The terms of series (21) are calculated recurrently by substitution of (21) into Eq. (17) with the use of expressions (11) and (14):

$$z_0 = v^{-1}, \quad z_1 = 2z_0^3(\alpha A - \varphi(0,0))(1 - z_0^2 + 2Az_0^2)^{-1}, \quad \dots .$$

Expressions (11) and (14) for $\omega \to +\infty$ ($\operatorname{Im}\omega = 0$) imply the asymptotic relation

$$\operatorname{Re} K_F(\omega, k) = 2\omega^{-2}(A\alpha - \varphi(0,0)) + O(\omega^{-3}).$$

Therefore, according to relation (5), the inequality $A\alpha - \varphi(0,0) \geqslant 0$ should hold. Hence the inequality $z_1 \geqslant 0$ follows.

We represent the solution of Eq. (16) in the form

$$u_F(\omega, k) = a(J_1(\omega, k) + J_2(\omega, k)),$$

$$J_1(\omega, k) = \Delta(\omega) 2k_1(\omega)(k^2 - k_1(\omega)^2)^{-1}, \quad J_2(\omega, k) = 1/\Psi(\omega, k) - J_1(\omega, k),$$

where $\Delta(\omega)^{-1} = (\partial\Psi/\partial k)\Big|_{k=k_1(\omega)}$.

Let us study the perturbation front structure at a fixed point $x > 0$, i. e., the dependence of the solution on the parameter $\tau = t - xv^{-1}$ for small values of this parameter. The function $J_2(\omega, k)$ is holomorphic in the complex tube T_1, and therefore, its Fourier preimage vanishes outside the cone C_1. This means that for sufficiently small τ, we have

$$u(t, x) = \frac{a}{(2\pi)^2} \int \exp(i(\omega t + kx)) J_1(\omega, k)\, d\omega\, dk,$$

where the integral with respect to k is calculated by the residue theorem. As a result, we have

$$u(t, x) = \frac{ai}{2\pi} \int \exp(i(\omega t + k_1(\omega)x)) \Delta(\omega)\, d\omega. \tag{22}$$

Using Eq. (21), for the integrand in (22), we obtain the asymptotic series

$$\exp(i(\omega t + k_1(\omega)x))\Delta(\omega) = -i \exp(i\omega\tau) \sum_{n=0}^{+\infty} X_n(x)(i\omega)^{-n},$$

$$X_0(x) = Az_0(1 - z_0^2)\exp(-z_1 x)(1 - z_0^2 + 2Az_0^2)^{-1}, \ \dots . \tag{23}$$

To calculate the asymptotics of function (22) for small τ on the basis of the asymptotic series (23) we use the method proposed in [4]. We specify a positive integer N and an arbitrary sequence of mutually different positive numbers y_j, $j = 0, \dots, N+1$. We introduce an auxiliary function

$$F_1(\omega, x) = b_{-1} + \sum_{j=0}^{N+1} b_j (i\omega + y_j)^{-1},$$

where the coefficients b_j are found from the system of linear equations

$$b_{-1} = X_0(x), \qquad \sum_{j=0}^{N+1} b_j y_j^k = (-1)^k X_{k+1}, \quad k = 0, \ldots, N. \tag{24}$$

System (24) always has a unique solution. Next, we introduce the functions

$$F_2(\omega, x) = \exp(ik_1(\omega)x)\Delta(\omega) + i\exp(-i\omega x v^{-1})F_1(\omega, x),$$

$$u_1(t, x) = \frac{a}{2\pi}\int \exp(i\omega\tau)F_1(\omega, x)\,d\omega, \tag{25}$$

$$u_2(t, x) = u(t, x) - u_1(t, x) = \frac{ai}{2\pi}\int \exp(i\omega t)F_2(\omega, x)\,d\omega.$$

Since $F_2(\omega) = O(\omega^{-(N+2)})$, which follows from the definitions, therefore, the function $u_2(t, x)$ is at least N times differentiable with respect to time and vanishes for $\tau < 0$. Thus, $u_2(t, x) = o(\tau^N)$. By the residue theorem, integral (25) is calculated explicitly:

$$u_1(t, x) = a\Big(b_{-1}\delta(\tau) + \theta(\tau)\sum_{j=0}^{N} b_j \exp(-y_j\tau)\Big).$$

Taking into account the above results, we conclude that the resultant formula describing the behavior of the function $u(t, x)$ in the vicinity of the front has the form

$$u(t, x) = a\Big(X_0\delta(\tau) + \theta(\tau)\sum_{j=0}^{N} X_{j+1}\,\frac{\tau^j}{j!} + o(\tau^N)\Big). \tag{26}$$

Formula (26) together with the calculation of the front propagation speed v is the main result of the present paper. It allows one to relate the perturbation front structure in transport processes to spatial–temporal nonlocality and determine the characteristics of the kernel $K(t, x)$ whose observation in other processes is difficult. Thus, experimental determination of v and the front structure can be used to refine the analytical structure of the kernel on the whole.

Formula (26) was derived for a point instantaneous source. In the case of a point source with intensity depending on time, an expression for perturbations in the vicinity of the front can be derived by convolving expression (26) with the source function.

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